VOLUME 79 SEPARATE No. 249

PROCEEDINGS

AMERICAN SOCIETY
OF
CIVIL ENGINEERS

AUGUST, 1953

A METHOD FOR CALCULATING STRESSES IN RIGID FRAME CORNERS

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STRUCTURAL DIVISION

{Discussion open until December 1, 1953}

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Printed in the United States of America

Headquarters of the Society
33 W. 39th St.
New York 18, N. Y.

PRICE $0.50 PER COPY

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This paper was published at 1745 S. State Street, Ann Arbor, Mich., by the American Society of Civil Engineers. Editorial and General Offices are at 33 West Thirty-ninth Street, New York 18, N. Y.

A METHOD FOR CALCULATING STRESSES IN RIGID FRAME CORNERS

Harvey C. Olander[1]/

INTRODUCTION

The purpose of this paper is to bring to the designer a simple method, based on the theory of a wedge, for calculating stresses in the corners of rigid frames where the extreme fibres are not parallel. The general idea was suggested some time ago in a paper by Wm. R. Osgood[2]/ and has been used in other solutions of this problem, although in more complicated form. However, it does not seem to be in general use, since discussions of the problem often appear, based on the erroneous assumption that the ordinary beam theory still holds on straight sections across members with nonparallel flanges. That these assumptions may be considerably in error, depending on the shape, was brought out in Mr. Osgood's article. Probably the use of the wedge theory in solving this problem would have come into more general use if the equations were simpler and the analyses made to follow a procedure similar to that used for beams with parallel flanges. Therefore, a method is proposed here based on an approximate solution for wedge-shaped beams using the familiar beam formulas; and in order to bring out the accuracy realized by the method, calculated stresses will be compared to measured stresses obtained from a number of photoelastic tests.

Procedure

The method is simply to take circular sections that cut the extreme fibres at right angles, such as Section AB, Figure 1a; develop the section as shown in Figure 1b and obtain the area A and moment of inertia I of the developed section. Next, resolve all forces to the right of Section AB into the forces P_o, V_o, and M_o about point o, the center of the arc. P_o shall pass through the center of gravity of Section AB, V_o shall be normal to P_o, and M_o is the moment of the forces about 0. Now with these forces, the stresses on Section AB are calculated as for an ordinary beam, except the shear will be determined from M_o. The total shear on Section AB is equal to $V = \frac{M_o}{r}$; the unit shear along Section AB = $\tau_{re} = \frac{VQ}{Ib}$. The stresses normal to the section = $\sigma_r = P_o/A \pm Mc/I$, where $M = M_o + V_o r$. M is simply the moment of the forces to the right of the section about the center of gravity, point X. It may seem unnecessary to first find V_o and M_o, then find their moments about point X, but since it is necessary to determine M_o to calculate the shear and P_o to find P_o/A, it is usually just as easy to resolve all the forces about point 0

1/ Engineer, Canals Branch, Design and Construction Division, Bureau of Reclamation, Denver, Colorado.

2/ Theory of Flexure for Beams with Nonparallel Extreme Fibres by Wm. R. Osgood, Trans. A.S.M.E., 1939.

Following this procedure, we may take circular sections as we please across the corner so long as they cut the extreme fibres at right angles, and the sides of the wedges considered are therefore tangent to the extreme fibres.

This is an approximate solution for the wedge-shaped beams and its accuracy varies according to the central angle β, being more accurate for the smaller values of β. In general, angles of β greater than 45° will not have to be considered, and for $\beta = 45^{\circ}$ the approximate solutions give results within 5 percent of the more rigorous solutions, and even greater accuracy for built-up members and for combinations of loads. The formulas for wedges given in Osgood's paper are repeated in the appendix.

The question will probably arise as to why the force V_o does not contribute to the shear on the circular sections, but a consideration of the forces as shown in Figure 2b will clear this up. The stresses σ_r all intersect at the center 0 and therefore the resultant of these stresses must pass through 0; further, the resultant is equal and opposite to V_o. Therefore, the stresses σ_r balance V_o completely and no tangential forces, τ_{re}, can be present.

The stresses resulting from M_o are both shear, τ_{re} and normal stresses, σ_r, Figure 2c. The resultant of τ_{re} is a vertical force equal and opposite to the resultant of σ_r, but these resultants are applied some distance, X, apart, forming a couple which balances M_o. For example, using the equations in the appendix for a section of uniform thickness, the resultants of σ_r and τ_{re} are equal to $\frac{8M_o \sin^3 \beta/2}{3r(\sin\beta - \beta\cos\beta)}$ but of opposite signs, and $X = \frac{3r(\sin\beta - \beta\cos\beta)}{8\sin^3 \beta/2}$.

For wedges loaded with concentrated forces at the apex, as in Figure 1a, the stresses σ_e, normal to σ_r, are equal to zero.

It should be pointed out that the proposed method for analyzing a corner such as that in Figure 1, neglects the effects of two principal factors in order to simplify the solution. This is necessary regardless of whether the solution of the stresses on the curved sections is made by the more rigorous or approximate formulas for wedges. The first factor is that the wedge is not loaded as shown by P_o, V_o, and M_o, but is loaded in some manner along the side 0A. Second, the shape of the wedge changes to the left of Section AB. There is a gradual change on the inside edge due to the curvature and at the corner there is an abrupt change in the outer edge. The effects of these factors on the results can best be demonstrated by comparing calculated stresses with measured stresses obtained from photoelastic tests.

In the following figures, the measured stresses are plotted as solid lines and those calculated by the approximate method as dotted lines. There will be some error present in the test results so that some judgment must be used, but by comparing the different figures an average discrepancy between measured and calculated stresses may be found for the different sections. It will be seen from the figures that results are good and the discrepancies occur where they would be expected.

In Figure 3 agreement of measured and calculated stresses is very good, except in regions near the vicinity of the sharp breaks in section. Ordinarily these stress concentrations are not critical since they extend only over a small area and are of secondary nature. In a corner made up of web and flanges, it would be well to add stiffeners at the breaks

to support the outstanding legs of the flanges.

In Figures 4 to 7 the inside radius has been varied from 5 inches to 2 inches. Figure 8 is the same shape as 7 except the sharp outside corner has been rounded off. Agreement is good along the outside edges of all corners. The greatest discrepancies occur on the inside edges toward the center of the arc, and are greatest where the internal angle of the wedge section is 45^{o}. The average agreement is within 8 percent up to internal angles of 30^{o}. It should be pointed out, however, that this is beyond the region where maximum stresses occur so that the error at point A is usually of no importance. Further, Sections AB in Figures 4 to 7 indicate the overstress is only near the extreme fibre and therefore does not extend over a large area.

Sections were taken as indicated. On the curved corners they occur at intervals having internal angles equal to 15^{o}, 30^{o}, and 45^{o}

The circled points indicated on Sections AB, Figures 4 to 7, were obtained by further refinement of the method or better approximations of the loads on the wedges. These will be explained later.

The photoelastic tests were made only on members of uniform thickness. Two tests3/4/ known to the writer were available for built-up members where a comparison could be made of measured stresses with those obtained using the approximate method, and are shown in Figures 9 and 10. The results for these tests show the same general agreement between measured and calculated stress around the corner as the photoelastic tests. The agreement is good and within the ordinary accuracy to be expected between design calculations and actual tests.

Earlier, two factors were mentioned that were neglected in order to simplify the problem. Of the two, that of replacing the loads along the wedge surface by the forces P_o, V_o, and M_o appears to influence the results more than the other. By St. Venant's principle this approximation would have little influence on the stresses on the outside edge, such as those at point B on Section AB, Figure 11, but would have the greatest effect at A on the inside, and it is significant that the greatest discrepancies occur along the inside edge. The change in section to the left of AB does not appear to influence the results for extreme fibre stresses much, since in regions where the local effects of the applied loads are small, agreement of measured and calculated stresses is good. For example, they agree very well along the outer edges everywhere in all tests, and in Figure 3 where the inside edges are straight, agreement is good on both edges even within close proximity of the sharp breaks.

To further substantiate these observations, an attempt was made to load the wedge A0B, Figure 11, along the side 0A, in a manner that would show the local effect of these loads on the Sections AB. In doing so, the stresses indicated by the circled points on Sections AB, Figures 4 to 7, were obtained which show good agreement and seem to account for the bulk of the discrepancies between measured and calculated stresses.

3/ An Investigation of Steel Rigid Frames by Inge Lyse and W. E. Black, Proc. A.S.C.E., November 1940.

4/ Strength of Riveted Steel Rigid Frame Having a Curved Inner Flange, by Ambrose H. Stang, Martin Greenspan, Wm. R. Osgood, N. B. St. Research Paper RP 1161.

Consider the wedge A0B of Figure 11. It is loaded along the face 0A by the stresses that exist on Section 0A. These stresses will be tangential and normal and somewhat of the nature shown in Figure 11a, and always equal to zero at point A. The local effect of the stresses along A0 on Section AB diminishes as we go farther away from A, so that somewhere, say beyond point X, the manner in which the stresses are applied to the wedge is not important and therefore those forces from X to 0 may be replaced with concentrated forces at point 0. In other words, to get better results on AB only those stresses from A to X need to be applied in the manner they exist along this distance. The total tangential and normal stresses from A to X can be found going back to some section as CD, and balancing forces T and N on the element ACX, Figure 11b, assuming, of course, that the stresses on CD have been previously determined. The normal stresses are small from A to X and were found to have small influence; therefore, only the tangential stresses from A to X were applied as such. Further, to simplify the work, they were assumed to have a triangular distribution as shown in Figure 11c. Finally, these stresses and the external forces were made to balance about point 0 by adjusting P'_0 and V'_0; then with the system of forces shown in Figure 11c, the stresses were calculated. As mentioned earlier these approximations yielded results which are in better agreement with measured stresses on Sections AB and the general shape of the stress distribution curves obtained across the sections has the same shape as the measured stresses, both σ_r and τ_{re}.

It should be pointed out that this attempt to correct the stresses on Sections AB, Figures 4 to 7, was merely done as a matter of interest and is not recommended as a practical method. It is still approximate and in order to improve it, more elaborate methods would have to be developed to load the wedge with the stresses as they exist along the side 0A. Even if this were possible, the work could hardly be justified since the stresses in question are not critical from a design standpoint. A reasonable approximation of the stress at point A can be made in a simple manner, as will be shown later.

It was pointed out earlier that the results obtained on the outer straight edges are very good and the inner curved edge may be calculated by this method to points where the internal angle of the wedge is 30^o or less with an expected accuracy within 10 percent. At the center of the curved edge where the internal angle is 45^o, point A, Figure 12, the greatest discrepancies occur, but calculations show that for the ordinary members used in structures and for any combination of loads, the stresses are not critical at this point and therefore not important. In calculating stresses at point A, it will be found that the stresses at A from Sections AB and AJ do not always coincide, depending on the type of loading. For the problems previously considered, they deviated as indicated by the two circled points at A, Figure 12b. However, knowing the stresses at points K and C, and considering the results of the photoelastic tests, it is a simple matter to estimate the stress at A by considering the general shape of the curve to either side of K and C. We may therefore interpolate the stresses similar to the dotted line shown in Figure 12b, or use an average of the stresses at K and C.

The method may also be used in similar problems, as for example, the bracket shown in Figure 13.

Two centers were used for taking circular sections. Those sections shown by the dashed lines have the center 0_1, and other sections such as A-D have the center 0_2.

The σ_x stresses on Section 0_2-C are shown for the load in position (1). There are also shear stresses τ_{xy} and stresses σ_y present on this section, but they are not shown. σ_x, τ_{xy}, and σ_y are determined at any point along Section 0_2-C from the stresses σ_r and τ_{re} of a circular section through that point.

To determine the stress atpoint B on Section A-D, the wedge 0_2AD must be considered, which is loaded along the side 0_2-A by stresses σ_x and τ_{xy}. Since B is some distance away from side 0_2-A, we may consider the wedge 0_2AD to be loaded with concentrated forces, P_o, V_o, and M_o at 0_2, obtained from resolving the stresses σ_x and τ_{xy} from 0_2 to A about 0_2. The load P to the left of Section 0_2-C is not to be used in this case to determine the concentrated forces at 0_2, but when the wedge becomes large enough that the side 0_2-A is equal to or greater than 0_2-C, P may be used because it will then be equal to the stresses σ_x and τ_{xy} from 0_2 to C, resolved about 0_2.

σ_x stresses on Section Y-Y are shown in Figure 13 for two positions of the load which were calculated in this manner by taking several circular sections like A-D and finding stresses at points such as B along the line Y-Y. Formulas and stress curves for straight sections like Y-Y which simplify this part of the work are given in the appendix.

It should be cautioned that in a problem of this type, a free surface in the near vicinity of Section Y-Y will have considerable effect on the results.

Summary

The method proposed is simple, easy to remember, and will give results within the accuracy usually obtained in design. It determines the extreme fibre stresses directly and simplifies the problem, since by taking circular sections only two types of stress are involved, namely σ_r and τ_{re}.

Stress calculations on the sections as shown in Figure 12 give sufficient information to design the corner. It is recommended to work from both sides as far into the corner as Sections AB and AJ. The stresses obtained on the outer edge will be nearly correct for all these sections and equal to zero at the corner. On the inside curved edge, the stresses obtained as far into the corner as points C and K may be used and the stress at point A interpolated as previously described.

The shear stresses have a tendency to shift toward the outside as shown in Section AB, Figures 4 and 5. However, maximum shear stresses are usually at the small sections where the parabolic distribution holds. In general, the approximate results obtained for shear on sections with an internal angle of $22\text{-}1/2^o$ or less are satisfactory.

Acknowledgements

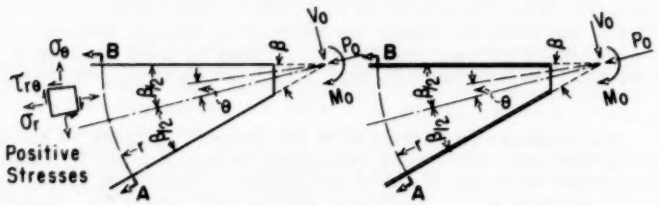
The writer is indebted to Engineers Robert Sailer, Carl N. Zangar, and H. Boyd Phillips of the Bureau of Reclamation, and Dr. C. K. Wang of the University of Colorado for their very helpful suggestions. The tests referred to were performed in the photoelastic laboratory of the Bureau of Reclamation, Denver, Colorado.

Appendix

The following formulas were used in the studies presented and are given here for their possible usefulness in other problems:

I. Wedges loaded with concentrated forces at the apex [5]

Wedges Loaded with Concentrated Forces at Apex



Uniform thickness | Symmetrical built-up section

Uniform thickness, t = 1

$$\sigma_r = \frac{P_o \cos\theta}{r(\beta/2 + 1/2 \sin\beta)}, \quad \sigma_\theta = 0, \quad \tau_{r\theta} = 0$$

$$= \frac{V_o \sin\theta}{r(\beta/2 - 1/2 \sin\beta)}, \quad = 0, \quad = 0$$

$$= \frac{4M_o \sin\theta \cos\theta}{r^2(\sin\beta - \beta\cos\beta)}, \quad = 0, \quad = \frac{-M_o(\cos 2\theta - \cos\beta)}{r^2(\sin\beta - \beta\cos\beta)}$$

Symmetrical built-up section

$$\sigma_r = \frac{P_o \cos\theta}{tr(\beta/2 + \sin\beta/2 \cos\beta/2) + 2A\cos^2\beta/2}, \quad \sigma_\theta = 0, \quad \tau_{r\theta} = 0$$

$$= \frac{V_o \sin\theta}{tr(\beta/2 - \sin\beta/2 \cos\beta/2) + 2A\sin^2\beta/2}, \quad = 0, \quad = 0$$

$$= \frac{2M_o}{r[tr(1 - \beta\cot\beta) + 2A\beta]}, \quad = 0, \quad = \frac{M_o[tr(\cos 2\theta - \cos\beta) + 2A\sin\beta]}{tr^2[tr(1 - \beta\cos\beta) + 2A\beta]}$$

where A = area of each flange and t = thickness of web.

II. The following formulas for wedges, loaded along one face, were developed from the general equations [6]. Those for uniform normal forces are also repeated here for convenience of reference.

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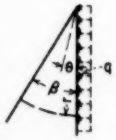
[5] See footnote 2, page 1

[6] Theory of Elasticity by S. Timoshenko

Uniform tangential load



$$\sigma_r = q[-2\theta - C(2\theta + \sin 2\theta) + D(1+\cos 2\theta)]$$

$$\tau_{r\theta} = q[1 + C(1-\cos 2\theta) - D\sin 2\theta]$$

$$\sigma_\theta = q[-2\theta - C(2\theta - \sin 2\theta) + D(1-\cos 2\theta)]$$

$$C = \frac{2\beta - \tan\beta}{2(\tan\beta - \beta)}$$

$$D = \frac{\beta\tan\beta - \beta\operatorname{ctn}\beta + 1}{2(\tan\beta - \beta)}$$

Triangular tangential load

$$\sigma_r = q'r[-\sin\theta + \operatorname{ctn}\beta\cos\theta(3\cos 2\theta - 1) - F\cos\theta\sin 2\theta]$$

$$\tau_{r\theta} = q'r[\cos\theta - \operatorname{ctn}\beta\sin\theta(3\cos 2\theta + 1) + F\sin\theta\sin 2\theta]$$

$$\sigma_\theta = 3q'r[\operatorname{ctn}\beta\sin\theta\sin 2\theta - \frac{\operatorname{ctn}^2\beta}{2}\sin\theta(1-\cos 2\theta) - \tfrac{1}{2}\cos\theta\sin 2\theta]$$

$$F = \frac{3(\operatorname{ctn}^2\beta - 1)}{2}$$

Parabolic tangential load

$$\sigma_r = 2q"r^2[G\cos 4\theta + \tfrac{1}{2}(H+1)\sin 4\theta]$$

$$\tau_{r\theta} = q"r^2[G(\sin 2\theta - 2\sin 4\theta) - H(\cos 2\theta - \cos 4\theta) + \cos 4\theta]$$

$$\sigma_\theta = 2q"r^2[G(\cos 2\theta - \cos 4\theta) + H(\sin 2\theta - \tfrac{1}{2}\sin 4\theta) - \tfrac{1}{2}\sin 4\theta]$$

$$G = \frac{2\cos^3\beta}{\sin\beta(1+2\cos^2\beta)}$$

$$H = \frac{-1+2\cos^2\beta - 4\cos^4\beta}{2\sin^2\beta(1+2\cos^2\beta)}$$

Uniform normal loads

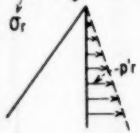
$$\sigma_r = p[1 + A(2\theta + \sin 2\theta) - B(1+\cos 2\theta)]$$

$$\tau_{r\theta} = p[-A(1-\cos 2\theta) + B\sin 2\theta]$$

$$\sigma_\theta = p[1 + A(2\theta - \sin 2\theta) - B(1-\cos 2\theta)]$$

$$A = \frac{1}{2(\tan\beta - \beta)}, \quad B = A\tan\beta$$

Triangular normal loads



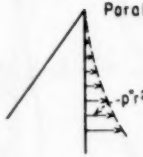
$$\sigma_r = p'r[\tfrac{1}{3}(1+E)\cos\theta - E\cos\theta\cos 2\theta + \operatorname{ctn}^3\beta\cos\theta\sin 2\theta]$$

$$\tau_{r\theta} = p'r[\tfrac{1}{3}(1+E)\sin\theta + E\sin\theta\cos 2\theta - \operatorname{ctn}^3\beta\sin\theta\sin 2\theta]$$

$$\sigma_\theta = p'r[\cos\theta - E\sin\theta\sin 2\theta + \operatorname{ctn}^3\beta\sin\theta(1-\cos 2\theta)]$$

$$E = \frac{2\cos^2\beta + 1}{2\sin^2\beta}$$

Parabolic normal loads



$$\sigma_r = \frac{-p"r^2}{2}[(2-J)\cos 4\theta - \frac{K}{2}\sin 4\theta]$$

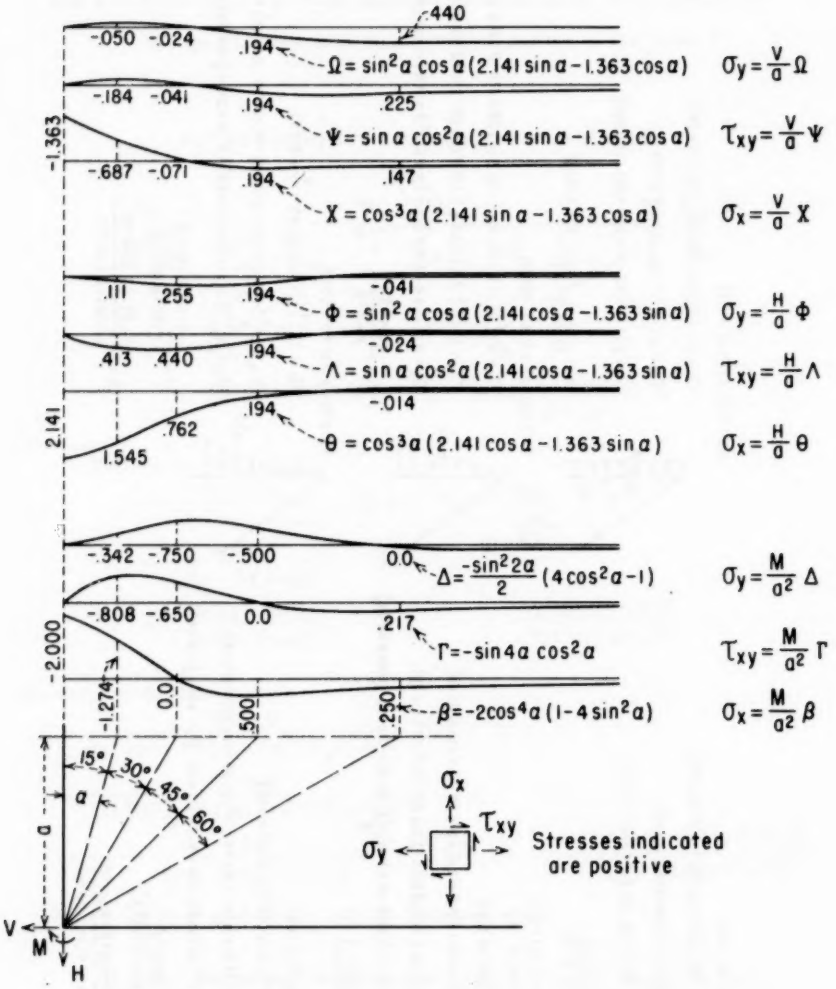
$$\tau_{r\theta} = \frac{p"r^2}{4}[J(\sin 2\theta - 2\sin 4\theta) - K(\cos 2\theta - \cos 4\theta) + 4\sin 4\theta]$$

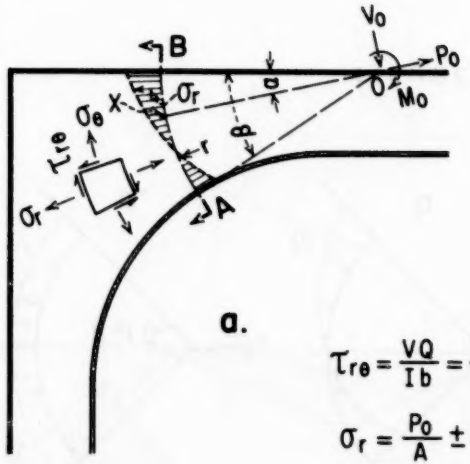
$$\sigma_\theta = \frac{p"r^2}{2}[J(\cos 2\theta - \cos 4\theta) + K(\sin 2\theta - \tfrac{1}{2}\sin 4\theta) + 2\cos 4\theta]$$

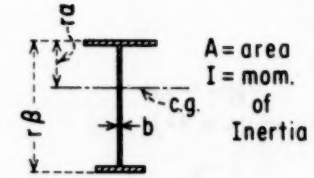
$$J = \frac{1 - 4\cos^2\beta\cos 2\beta}{1 - \cos^2\beta\cos 2\beta}$$

$$K = \frac{\operatorname{ctn}\beta(1+2\cos^2 2\beta)}{1 - \cos^2\beta\cos 2\beta}$$

III. Stresses in a massive block loaded with concentrated forces at the corner.







SECTION A-B
(Developed)

$$\tau_{r\theta} = \frac{VQ}{Ib} = \frac{M_o Q}{r\, Ib}$$

$$\sigma_r = \frac{P_o}{A} \pm \frac{Mc}{I} = \frac{P_o}{A} \pm (M_o + V_o r)\frac{C}{I}$$

$$\sigma_\theta = 0$$

FIGURE 1

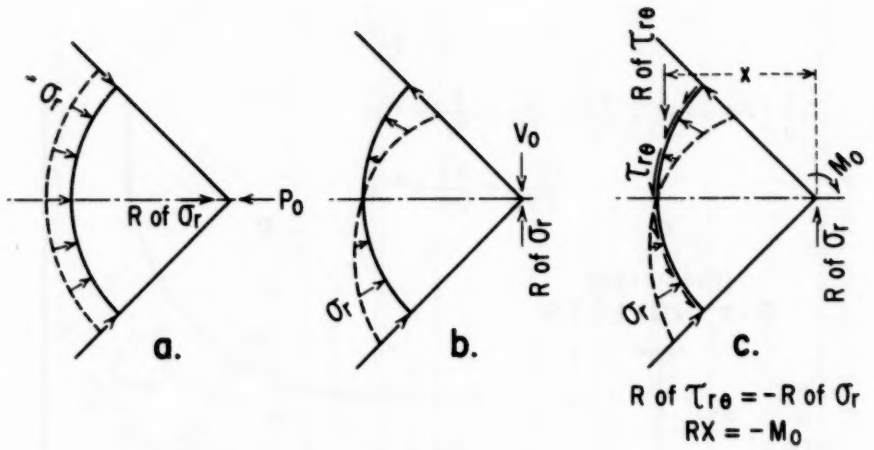

σ_r
R of σ_r
P_0
a.
V_0
R of σ_r
σ_r
b.
R of $\tau_{r\theta}$
x
$\tau_{r\theta}$
M_0
R of σ_r
σ_r
c.
R of $\tau_{r\theta} = -$R of σ_r
$RX = -M_0$

FIGURE 2

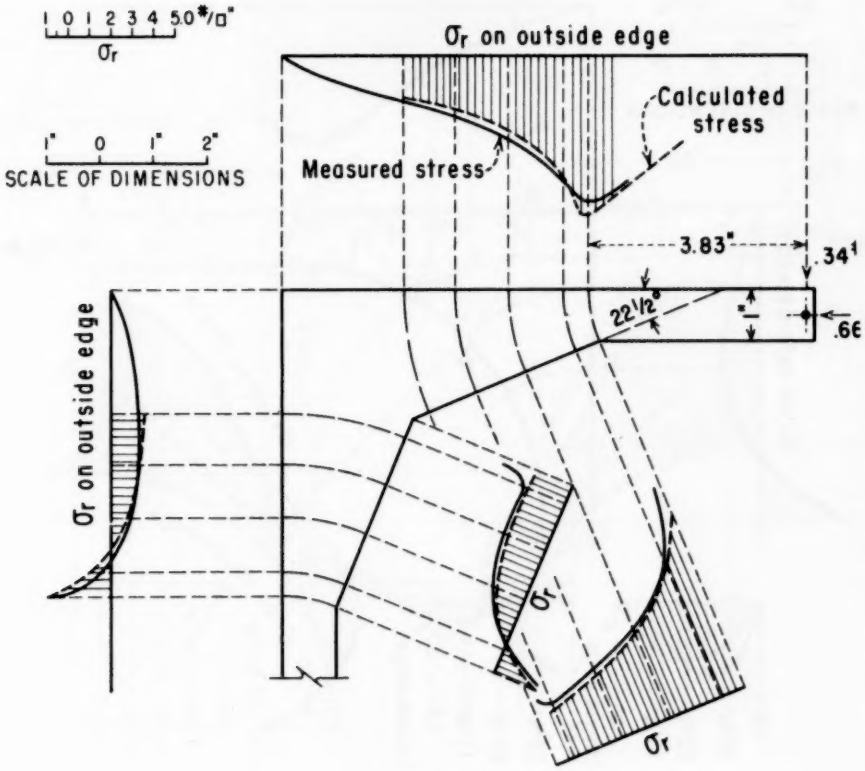
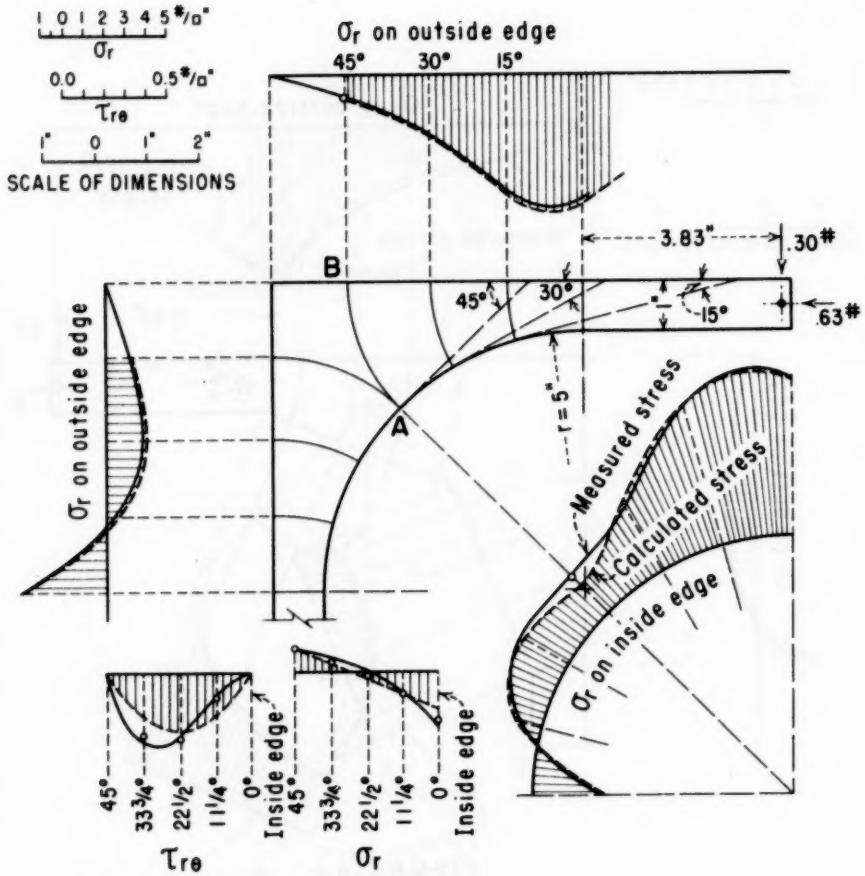
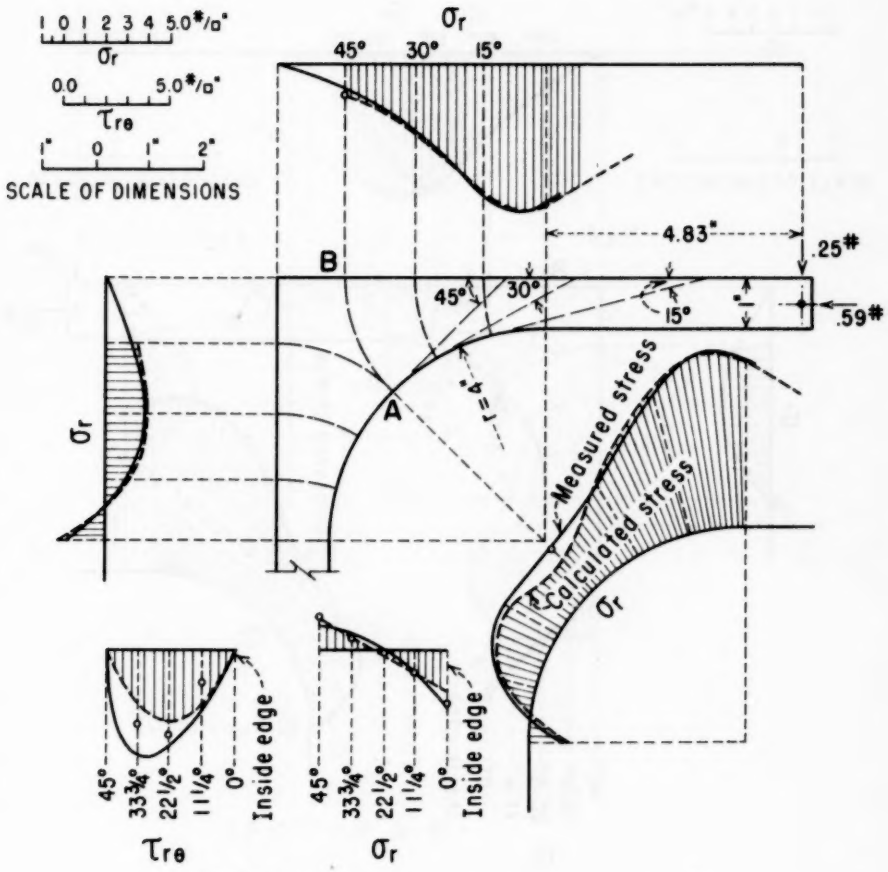

1 0 1 2 3 4 5 0 #/□"
σr
1" 0 1" 2"
SCALE OF DIMENSIONS
σr on outside edge
Calculated stress
Measured stress
3.83"
.34"
22 1/2°
.66"
σr on outside edge
σr
σr

FIGURE 3



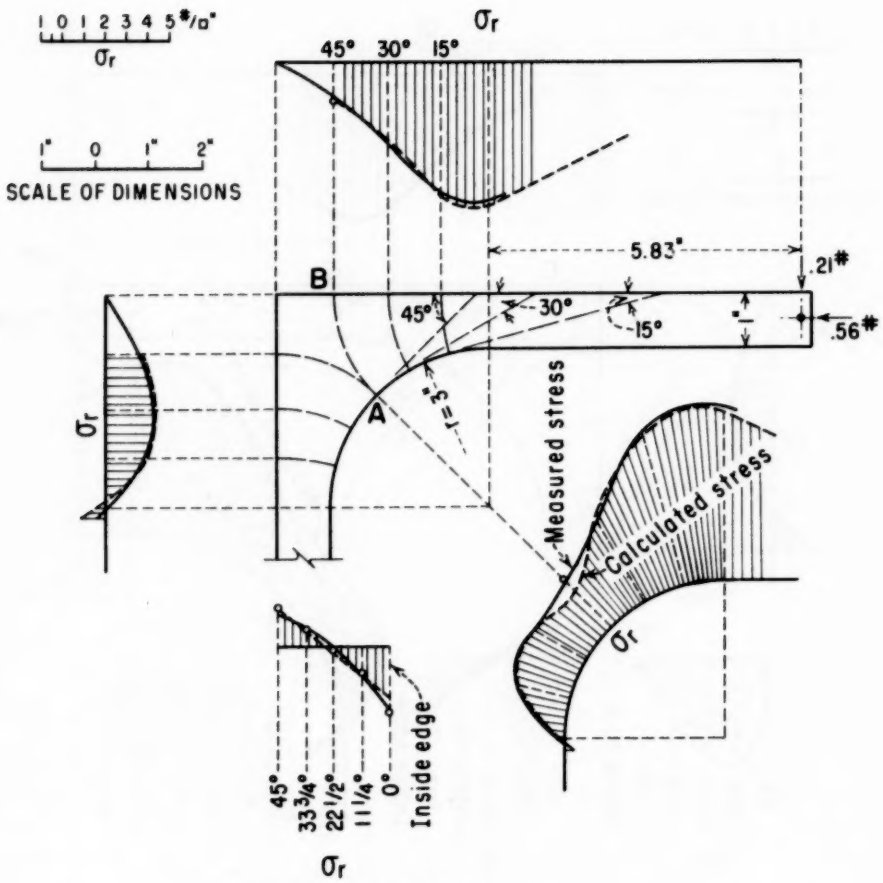
STRESSES ON SEC. A-B

FIGURE 4



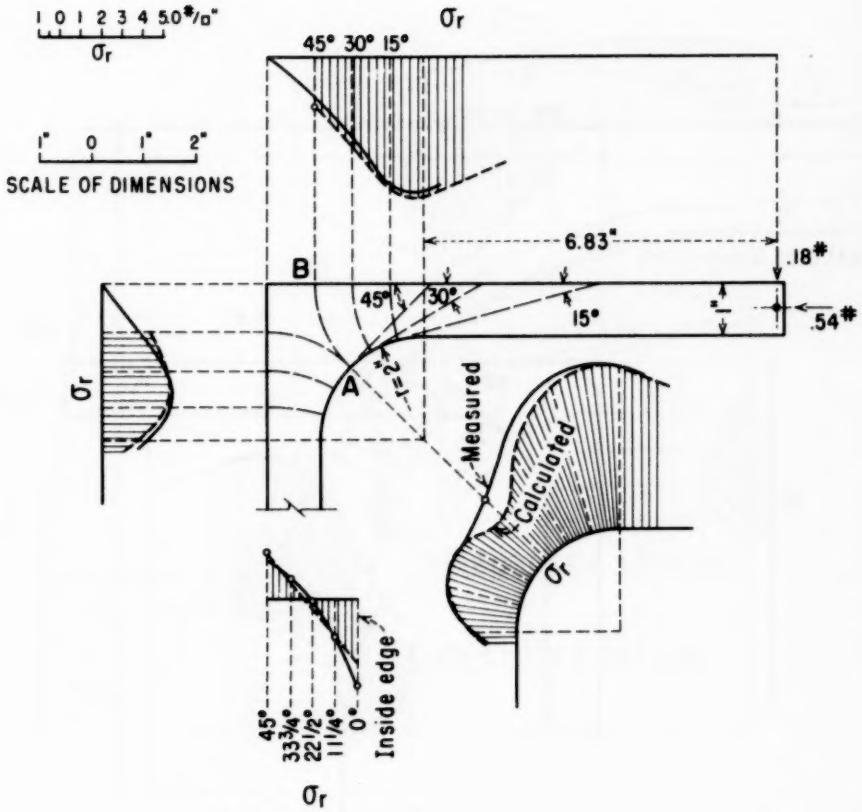
SECTION A-B

FIGURE 5



SECTION A-B

FIGURE 6



SECTION A-B

FIGURE 7

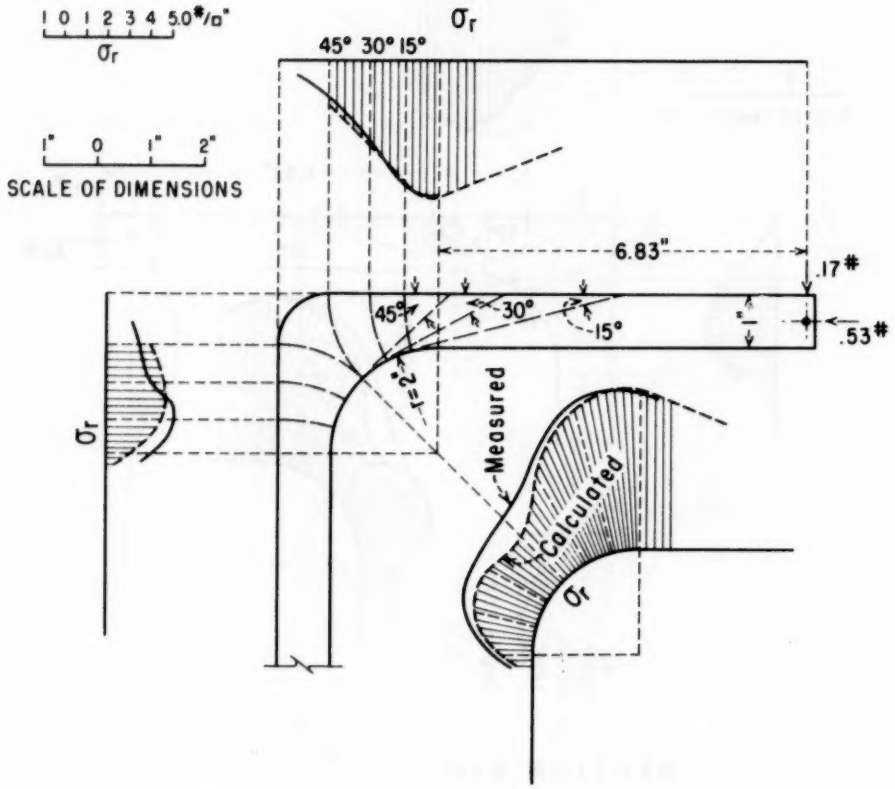

1 0 1 2 3 4 5.0 #/□"
σr
1" 0 1" 2"
SCALE OF DIMENSIONS
σr
45° 30° 15°
6.83"
.17#
.53#
45°
30°
15°
t=2"
Measured
Calculated
σr
σr

FIGURE 8

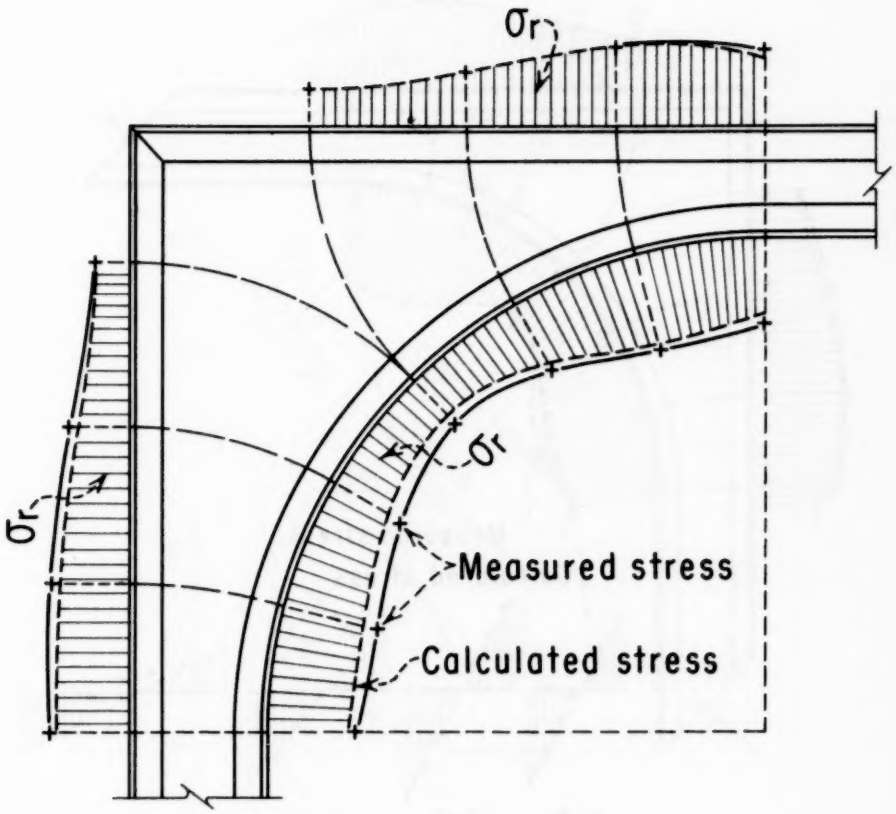


FIGURE 9

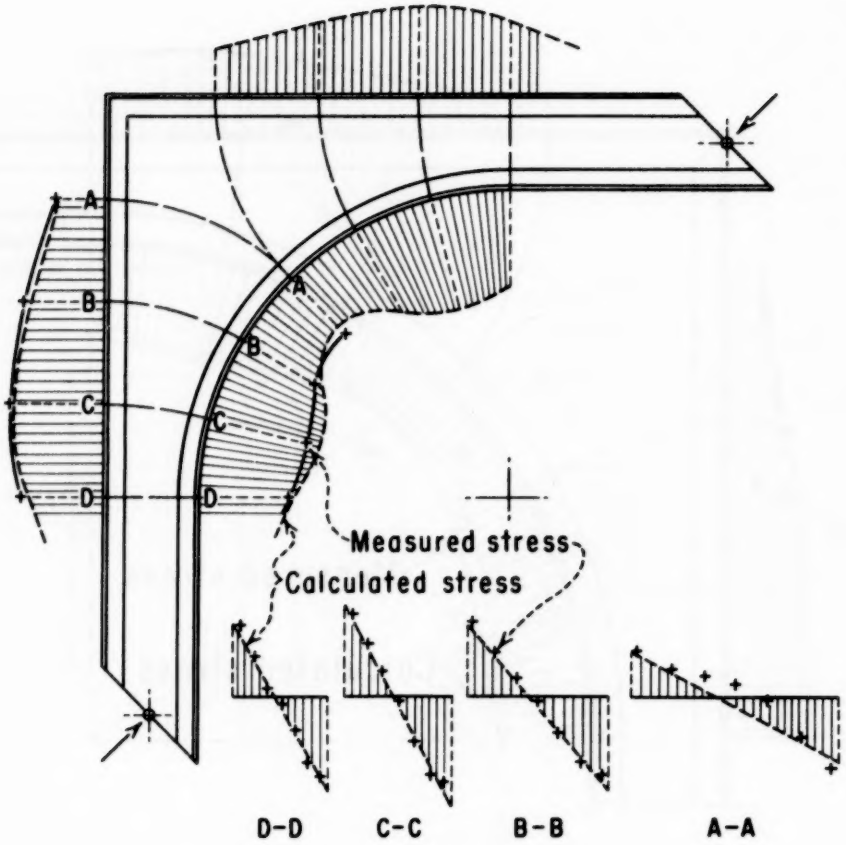


FIGURE 10

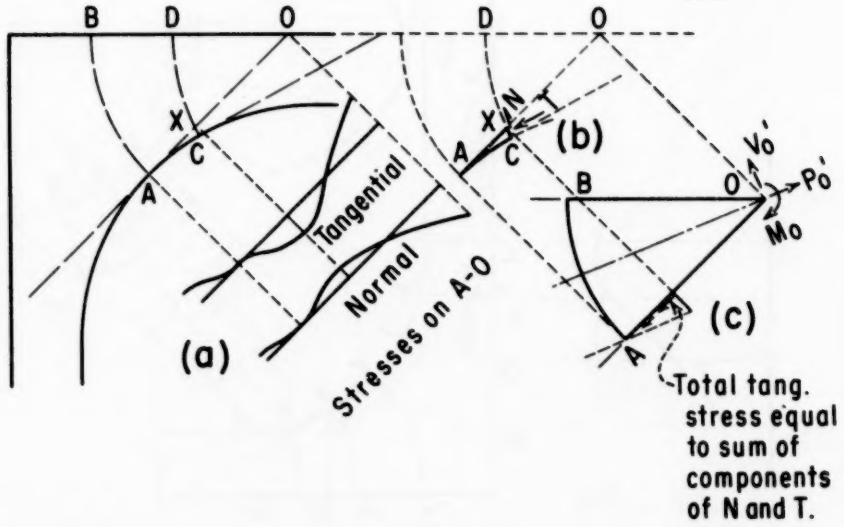

B
D
O
X
C
A
(a)
Tangential
Normal
Stresses on A-O
D
O
N
T
X
C
A
(b)
Vo'
Po'
B
O
Mo
(c)
A
Total tang. stress equal to sum of components of N and T.

FIGURE II

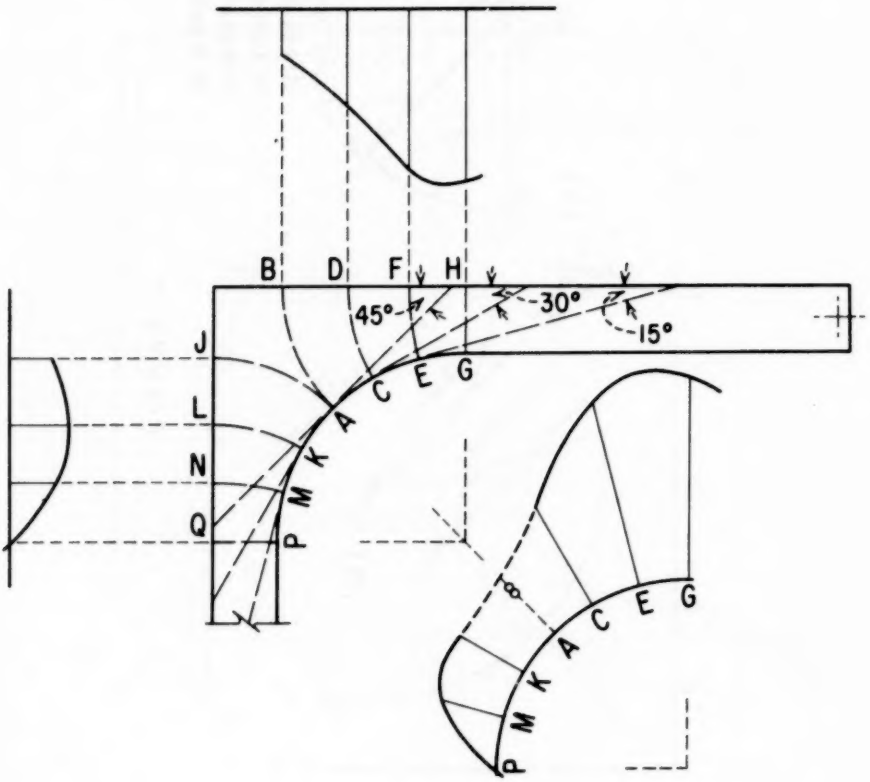


FIGURE 12

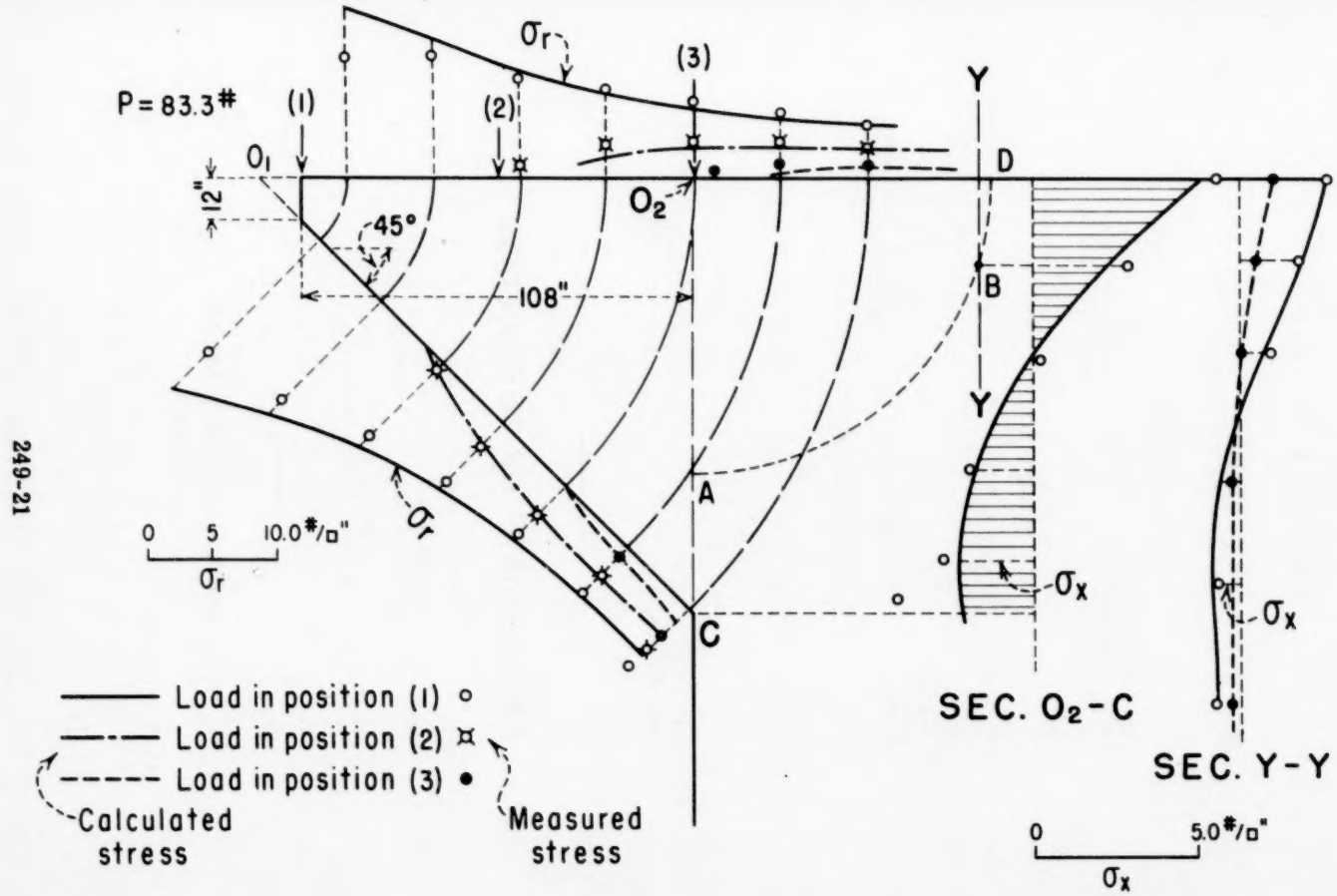


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FIGURE 13

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